

Composite Representation Invariants and Unoriented Topological String Amplitudes

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Abstract

Sinha and Vafa [1] had conjectured that the SO Chern-Simons gauge theory on S^3 must be dual to the closed A -model topological string on the orientifold of a resolved conifold. Though the Chern-Simons free energy could be rewritten in terms of the topological string amplitudes providing evidence for the conjecture, we needed a novel idea in the context of Wilson loop observables to extract cross-cap $c = 0, 1, 2$ topological amplitudes. Recent paper of Marino [2] based on the work of Morton and Ryder [3] has clearly shown that the composite representation placed on the knots and links plays a crucial role to rewrite the topological string cross-cap $c = 0$ amplitude. This enables extracting the unoriented cross-cap $c = 2$ topological amplitude. In this paper, we have explicitly worked out the composite invariants for some framed knots and links carrying composite representations in $U(N)$ Chern-Simons theory. We have verified generalised Rudolph's theorem, which relates composite invariants to the invariants in $SO(N)$ Chern-Simons theory, and also verified Marino's conjectures on the integrality properties of the topological string amplitudes. For some framed knots and links, we have tabulated the BPS integer invariants for cross-cap $c = 0$ and $c = 2$ giving the open-string topological amplitude on the orientifold of the resolved conifold.

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1 Introduction

We have seen interesting developments in the open string and closed string dualities during the last 12 years starting from the celebrated work of Maldacena [4]. Gopakumar and Vafa [5–7] conjectured open-closed duality in the topological string context. Gopakumar-Vafa conjecture states that the A -model open topological string theory on the deformed conifold, equivalent to the Chern-Simons gauge theory on S^3 [8], is dual to the closed string theory on a resolved conifold.

In ref. [5], it was shown that the free-energy expansion of $U(N)$ Chern-Simons field theory on S^3 at large N resembles A -model topological string theory amplitudes on the resolved conifold. This provided an evidence for the conjecture. Another piece of evidence at the level of observables was shown by Ooguri and Vafa [9] for the simplest Wilson loop observable (simple circle also called unknot) in Chern-Simons theory on S^3 . In particular, Ooguri-Vafa considered the expectation value of a scalar operator $\mathcal{Z}_{\mathcal{H}}(v)$ in the topological string theory corresponding to the simple circle in submanifold S^3 of the deformed conifold and showed its form in the resolved conifold background. From these results for unknot, Ooguri-Vafa conjectured on the form for $\mathcal{Z}_{\mathcal{H}}(v)$ for any knot or link in S^3 . For completeness and simplicity, we briefly present the form for knots:

$$\mathcal{F}_{\mathcal{H}}(v) = \ln \mathcal{Z}_{\mathcal{H}}(v) = \ln \left\{ \sum_R \mathcal{H}_R[\mathcal{K}] s_R(v) \right\} = \sum_{R,d} f_R(q^d, \lambda^d) s_R(v^d) \quad (1.1)$$

$$\text{where } f_R(q, \lambda) = \frac{1}{(q^{1/2} - q^{-1/2})} \sum_{Q,s} N_{R,Q,s} \lambda^Q q^s \quad (1.2)$$

Here $\mathcal{H}_R(\mathcal{K})$ are the $U(N)$ Chern-Simons invariants for a knot \mathcal{K} in S^3 carrying representation R and $s_R(v)$ are the Schur polynomials in variable v which represent $U(N)$ holonomy of the knot \mathcal{K} in the Lagrangian submanifold \mathcal{N} which intersects S^3 along the knot. $\mathcal{F}_{\mathcal{H}}(v)$ denotes the free-energy of the topological open-string partition function on the resolved conifold and $f_R(q, \lambda)$ are the $U(N)$ reformulated invariants. The conjecture states that the reformulated invariant must have the form (1.2) where $N_{R,Q,s}$ are integer coefficients.

Labastida-Marino [10] used group-theoretic techniques to rewrite the expectation value of the topological operators in terms of link invariants in $U(N)$ Chern-Simons field theory on S^3 . This group theoretic approach enabled verification of Ooguri-Vafa conjecture for many non-trivial knots [10–13]. Conversely, the Ooguri-Vafa conjecture led to a reformulation of Chern-Simons field theory invariants for knots and links giving new polynomial invariants (1.2). The integer coefficients of these new polynomial invariants have topological meaning accounting for BPS states in the string theory. The challenge still remains in obtaining such integers for non-trivial knots and links within topological string theory.

Another challenging question is to attempt similar duality conjectures between Chern-Simons gauge theories on three-manifolds other than S^3 and closed string theories. Invoking Gopakumar-Vafa conjecture and Ooguri-Vafa conjecture, it was possible to explicitly write the $U(N)$ Chern-Simons free-energy expansion at large N as a closed string theoretic expansion [14]. Surprisingly, the expansion resembled partition function of a closed string theory on a Calabi-Yau background with one kahler parameter. Unfortunately, the Chern-Simons free-energy expansion for other three-manifolds are not equivalent to the ‘t Hooft large N perturbative expansion around a classical solution [15]. In order to predict new duality conjectures, we need to extract the perturbative expansion around a classical solution from the free-energy.

For orbifolds of S^3 , which gives Lens space $\mathcal{L}[p, 1] \equiv S^3/Z_p$, it is believed that the Chern-Simons theory is dual to the A -model closed string theory on A_{p-1} fibred over P^1 Calabi-Yau background. It was Marino [16] who showed that the perturbative Chern-Simons theory on Lens space $\mathcal{L}[p, 1]$ can be given a matrix model description. Also, hermitian matrix model description of B -model topological strings [17] was shown to be equivalent to Marino’s matrix model using mirror symmetry [18]. It is still a challenging open problem to look for dual closed string description corresponding to $U(N)$ Chern-Simons theory on other three-manifolds.

The extension of these duality conjectures for other gauge groups like $SO(N)$ and $Sp(N)$ have also been studied. In particular, the free-energy expansion $F_{(CS)}^{(SO)}[S^3]$ of the Chern-Simons theory on S^3 based on SO gauge group was shown to be dual to A -model closed string theory on a orientifold of the resolved conifold background [1]. In particular, the string partition function Z for these orientifolding action must have two contributions:

$$F_{(CS)}^{(SO)}[S^3] = Z = \frac{1}{2}Z^{or} + Z^{(unor)} \quad (1.3)$$

where $Z^{(or)}$ is the untwisted contribution and $Z^{(unor)}$ is the twisted sector contribution. The untwisted contribution exactly matches the $U(N)$ Chern-Simons free energy on S^3 . Using the topological vertex as a tool, Bouchard et al [19,20] have determined unoriented closed string amplitude and unoriented open topological string amplitudes for a few orientifold toric geometry with or without D -branes.

In Ref. [21], the generalisation of Ooguri-Vafa conjecture for observables involving $SO(N)$ holonomy, different from the works of Bouchard et al [19,20], was studied. Similar to the $U(N)$ result (1.2), the coefficients of $SO(N)$ reformulated invariants are indeed integers.

Following Sinha-Vafa conjecture [1], the expectation value of the topological string operator (observables) $Z_{\mathcal{G}}(v)$ where \mathcal{G} represents $SO(N)$ knot invariants in Chern-Simons theory on S^3 and v represents the SO holonomy on the submanifold \mathcal{N} intersecting S^3 along a knot. It is expected that the free-energy of the open-string partition function on the orientifold of the

resolved conifold must also satisfy a relation similar to eqn.(1.3):

$$\mathcal{F}_{\mathcal{G}}(v) = \ln Z_{\mathcal{G}}(v) = \frac{1}{2}\mathcal{F}_{\mathcal{R}}^{(or)}(v) + \mathcal{F}^{(unor)}(v) . \quad (1.4)$$

where $\mathcal{F}_{\mathcal{R}}^{(or)}(v)$ is the oriented or untwisted sector contribution (also called cross-cap $c = 0$) and the twisted sector term $\mathcal{F}^{(unor)}(v)$ will have both cross-cap $c = 1$ and $c = 2$ contributions to the open topological string amplitudes. It was not clear [19, 20] as to how to obtain $\mathcal{F}_{\mathcal{R}}^{(or)}(v)$ in the orientifold theory using $U(N)$ Chern-Simons knot invariants. As a result, it was not possible to distinguish the topological amplitudes of cross-cap $c = 0$ from $c = 2$ contribution. However using parity argument in variable $\sqrt{\lambda}$, the cross-cap $c = 1$ topological amplitudes contribution could be obtained [19–21].

From the orientifolding action, Marino [2] has indicated that there must be a $U(N)$ composite representation (R, S) placed on the knot in S^3 and the oriented contribution must be rewritable as:

$$\mathcal{F}_{\mathcal{R}}^{(or)}(v) = \sum_{R, S} \mathcal{H}_{(R, S)}[\mathcal{K}] s_R(v) s_S(v) = \sum_R \mathcal{R}_R[\mathcal{K}] s_R(v) \quad (1.5)$$

where $s_R(v)$ and $s_S(v)$ are the Schur polynomials corresponding to the $U(N)$ holonomy in two Lagrangian submanifolds \mathcal{N}_{ϵ} and $\mathcal{N}_{-\epsilon}$ related by the orientifolding action. Here ϵ denotes the deformation parameter of the deformed conifold. The oriented invariant $\mathcal{R}_R[\mathcal{K}]$ can be obtained from composite invariants $\mathcal{H}_{(R, S)}[\mathcal{K}]$ using the properties of the Schur polynomials. Though we have so far discussed for knots, it is straightforward to generalise these arguments for any r -component link L .

In this paper, we explicitly evaluate the composite invariants $\mathcal{H}_{(R_1, S_1), (R_2, S_2), \dots, (R_r, S_r)}[L]$, in $U(N)$ Chern-Simons gauge theory for many framed knots and links L made of r component knots \mathcal{K}_{α} 's carrying composite representations (R_{α}, S_{α}) using the tools [22]. These composite invariants are polynomials in two variables q, λ . We find that the framing factor for the component knots of the links carrying composite representation requires a slightly modified choice of the $U(1)$ charge so that the composite invariants are polynomials in variables q and λ .

Comparing these invariants with $SO(N)$ Chern-Simons invariants $\mathcal{G}_{R_1, R_2, \dots, R_r}[L]$ [21] for link L whose components carry representations R_{α} 's which are also polynomials in two variables (q, λ) , we have verified the generalised Rudolph's theorem [3, 23]:

$$\frac{1}{2} \left[\mathcal{H}_{(R, R)}[\mathcal{K}] + \{\mathcal{G}_R[\mathcal{K}]\}^2 \right] = f(q) \sum_{n, p} a_{n, p} \lambda^{\frac{n}{2}} q^p , \quad (1.6)$$

for many framed knots \mathcal{K} carrying $R = \boxplus, \boxtimes, \square$. Here $f(q)$ is a function of q , $a_{n, p}$ are integers. In fact, the above relation between $U(N)$ composite invariants and $SO(N)$ invariants appears naturally from the integrality properties of the topological string amplitudes in the orientifold

geometry [2]. Using these composite representation invariants, we verified the integrality conjectures of Marino [2] for framed knots and framed two-component links. While submitting this paper, we came across a recent paper [24] where Marino's conjectures have been verified for standard framing torus knots and torus links which is a special case of our results.

The organisation of the paper is as follows. In section 2, we present composite framed knot and framed two-component link invariants in $U(N)$ Chern-Simons theory. In section 3, we briefly review Marino's conjectures on the reformulated invariants of the framed links in the orientifold resolved conifold. In section 4, we verify Marino's conjectures and tabulate the $c = 0, c = 2$ BPS integer coefficients for few examples. In the concluding section, we summarize the results obtained. In appendix A, we present $U(N)$ composite invariants for some framed knots and framed two-component links for some representations.

2 Chern-Simons Gauge theory and Composite Link invariants

Chern-Simons gauge theory on S^3 based on the gauge group G is described by the following action:

$$S = \frac{k}{4\pi} \int_{S^3} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (2.1)$$

where A is a gauge connection for compact semi-simple gauge group G and k is the coupling constant. The observables in this theory are Wilson loop operators:

$$W_{R_1, R_2, \dots, R_r}[L] = \prod_{\alpha=1}^r \text{Tr}_{R_\alpha} U[\mathcal{K}_\alpha] , \quad (2.2)$$

where $U[\mathcal{K}_\alpha] = P \left[\exp \oint_{\mathcal{K}_\alpha} A \right]$ denotes the holonomy of the gauge field A around the component knot \mathcal{K}_α of a r -component link L carrying representation R_α . The expectation value of these Wilson loop operators are the link invariants:

$$\langle W_{R_1, R_2, \dots, R_r}[L] \rangle(q, \lambda) = \frac{\int [\mathcal{D}A] e^{iS} W_{R_1, R_2, \dots, R_r}[L]}{\int [\mathcal{D}A] e^{iS}} , \quad (2.3)$$

These link invariants are polynomials in two variables

$$q = \exp \left(\frac{2\pi i}{k + C_v} \right) , \quad \lambda = q^{N+a} , \quad (2.4)$$

where C_v is the dual coxeter number of the gauge group G

$$C_v = \begin{cases} N & \text{for } G = SU(N) \\ N - 2 & \text{for } G = SO(N) \end{cases} \quad \text{and } a = \begin{cases} 0 & \text{for } G = SU(N) \\ -1 & \text{for } G = SO(N) \end{cases}$$

These link invariants can be computed using the following two inputs [22]:

- (i) Any link can be drawn as a closure or plat of braids,
 - (ii) The connection between Chern-Simons theory and the Wess-Zumino conformal field theory.
- We now define some quantities which will be useful later. The quantum dimension of a representation R with highest weight Λ_R is given by

$$\dim_q R = \prod_{\alpha > 0} \frac{[\alpha \cdot (\rho + \Lambda_R)]}{[\alpha \cdot \rho]} , \quad (2.5)$$

where α 's are the positive roots and ρ is the Weyl vector equal to the sum of the fundamental weights of the group G . The square bracket refers to the quantum number defined by

$$[x] = \frac{(q^{x/2} - q^{-x/2})}{(q^{1/2} - q^{-1/2})} . \quad (2.6)$$

The $SU(N)$ quadratic Casimir for representation R is given by

$$C_R = -\frac{\ell^2}{2N} + \kappa_R = -\frac{\ell^2}{2N} + \frac{1}{2} \left((N+a)\ell + \ell + \sum_i (l_i^2 - 2il_i) \right) . \quad (2.7)$$

Our interest is to obtain invariants of framed knots and framed links carrying representation $R_c \equiv (R, S)$ called composite representation in $U(N)$ Chern-Simons gauge theory so that Marino's conjectures on the topological amplitudes in the orientifold of resolved conifold geometry can be verified.

2.1 Composite Invariants in $U(N)$ Chern-Simons Gauge Theory

The composite representation, $R_c \equiv (R, S)$ labelled by a pair of Young diagram is defined as [2, 25–27]

$$R_c \equiv (R, S) = \sum_{U, V, W} (-1)^{\ell(U)} N_{UV}^R N_{TW}^S (V \times \bar{W}) , \quad (2.8)$$

where U, V, W are the representations of the group $U(N)$, $\ell(U)$ denotes the number of boxes in the Young diagram corresponding to U and N is the Littlewood-Richardson coefficient for multiplication of the Young diagrams.

If we take the simplest defining representation for $R = \square$ and $S = \bar{\square}$, then the composite representation $R_c = (\square, \bar{\square})$ derived from eqn. (2.8) will be the adjoint representation of $U(N)$. In terms of fundamental weights, the highest weight of R_c is $\Lambda^{(1)} + \Lambda^{(N-1)}$. Using the above eqn.(2.8), one can obtain the $SU(N)$ representation for any composite representation (R, S) and the corresponding highest weight will be $\Lambda_R + \Lambda_{\bar{S}}$ where Λ_R and $\Lambda_{\bar{S}}$ are the highest weights of representation R and conjugate representation \bar{S} respectively.

We will now explicitly evaluate the polynomials for various knots and links carrying the composite representation (R, S) in $U(N)$ Chern-Simons theory. For the simplest circle called unknot U_p with an arbitrary framing p , the composite invariant will be framing factor multiplying the quantum dimension of the composite representation (R, S) :

$$\mathcal{H}_{(R,S)}[U_p] = (-1)^{\ell p} q^{\frac{p\{n_{(R,S)}\}^2}{2}} q^{pC_{(R,S)}} \dim_q(R, S) . \quad (2.9)$$

where ℓ is the total number of boxes in the Young diagram for composite representation (R, S) , $C_{(R,S)}$ denotes the $SU(N)$ quadratic casimir (2.7) and $n_{(R,S)}$ represents the $U(1)$ charge for the composite representation (R, S) . Looking at the definition of the composite representation highest weight, we propose that the $U(1)$ charge $n_{(R,S)}$ must be the difference of $U(1)$ charges n_R and n_S of representation R and S :

$$n_{(R,S)} = |n_R - n_S| . \quad (2.10)$$

Earlier the $U(1)$ charges for $U(N)$ representations were chosen [13,14] such the $U(N)$ invariants are polynomials in two variables q, λ [13,14]. For representation R with $\ell(R)$ number of boxes in the Young diagram representation, the $U(1)$ charge n_R is

$$n_R = \frac{\ell(R)}{\sqrt{N}} . \quad (2.11)$$

Substituting the $U(1)$ charge (2.11) in eqn.(2.10), the unknot invariant (2.9) simplifies to

$$\mathcal{H}_{(R,S)}[U_p] = (-1)^{\ell p} q^{\kappa_R + \kappa_S} \dim_q(R, S) . \quad (2.12)$$

Our choice for the composite representation $U(1)$ charge (2.10) results in the simplified form for the framing factor. For knots carrying composite representation (R, S) , the framing factor in eqn.(2.12) involves only the sum of κ_R and κ_S (2.7).

Now, we can write the $U(N)$ framed knot invariants for torus knots $\mathcal{K}_{2m+1}^{(p)}$ of the type $(2, 2m+1)$ with framing p as follows:

$$\mathcal{H}_{(R,S)}[\mathcal{K}_{2m+1}^{(p)}](q, \lambda) = (-1)^{\ell p} q^{p(\kappa_R + \kappa_S)} \sum_{R_t} \dim_q R_t (\lambda_t)^{2m+1} , \quad (2.13)$$

where $R_t \in (R, S) \otimes (R, S)$ and λ_t is the braiding eigenvalue in standard framing ($p = 0$) for parallelly oriented strands:

$$\lambda_t = \epsilon_{R_t} q^{2C_{(R,S)} - C_{R_t}/2} , \quad (2.14)$$

where $\epsilon_{R_t} = \pm 1$ depending upon whether the representation R_t appears symmetrically or antisymmetrically with respect to the tensor product $(R, S) \otimes (R, S)$ in the $U(N)_k$ Wess-Zumino Witten model. Unlike the totally symmetric or totally antisymmetric representations,

the tensor product of composite representations does give multiplicities and hence determining ϵ_{R_t} is non-trivial.

We have fixed the sign of ϵ_{R_t} by equating the invariants of two knots which are equivalent. For example, unknot U_0 and the torus knot $\mathcal{K}_1^{(0)}$ are equivalent. Taking the difference of these two knot polynomials and equating the coefficients of every power of q to zero, we obtain the signs of ϵ_{R_t} uniquely. In appendix A, we have explicitly given all the irreducible representations R_t and the signs ϵ_{R_t} for some composite representations so that the composite invariants can be computed. This will be very useful for verifying Rudolph theorem and Marino's conjectures. We can also check Marino's conjecture for composite invariant for connected sum of two knots \mathcal{K}_1 and \mathcal{K}_2 defined as:

$$\mathcal{H}_{(R,S)}[\mathcal{K}_1 \# \mathcal{K}_2] = \left(\mathcal{H}_{(R,S)}[\mathcal{K}_1] \mathcal{H}_{(R,S)}[\mathcal{K}_2] \right) / \mathcal{H}_{(R,S)}[U_0] . \quad (2.15)$$

The $U(N)$ invariants for framed torus links of the type $(2, 2m)$ can also be written. For example, the $U(N)$ invariant for a Hopf link of type $(2, 2)$ with linking number -1 and framing numbers p_1 and p_2 on the component knots carrying representations (R_1, S_1) and (R_2, S_2) will be

$$\mathcal{H}_{(R_1, S_1), (R_2, S_2)}[H](q, \lambda) = (-1)^{\ell_1 p_1 + \ell_2 p_2} q^{p_1(\kappa_{R_1} + \kappa_{S_1}) + p_2(\kappa_{R_2} + \kappa_{S_2})} \times \\ q^{\ell k n_{(R_1, S_1)} n_{(R_2, S_2)}} \sum_{R_t} \dim_q R_t q^{C_{(R_1, S_1)} + C_{(R_2, S_2)} - C_{R_t}} , \quad (2.16)$$

where $\ell k = -1$ is the linking number between the two-components and $R_t \in (R_1, S_1) \otimes (R_2, S_2)$. We now explicitly evaluate the knot polynomials carrying the composite representation (\square, \square) in $U(N)$ Chern-Simons theory, for the knots upto five crossings. For the simplest composite representation (\square, \square) , which we denote by ρ_0 , the highest weight is $\Lambda^{(N-1)} + \Lambda^{(1)}$. The p -frame unknot invariant for this representation is

$$\mathcal{H}_{(\square, \square)}[U_p] = (-1)^{\ell p} \lambda^p (\dim_q \rho_0) = (-1)^{Np} \lambda^p [N-1][N+1] , \quad (2.17)$$

where rewriting the quantum numbers (2.6) will give the p -framed unknot invariant in variables $q, \lambda = q^N$. The highest weights for all the representations R_t 's obtained from $\rho_0 \otimes \rho_0$ and their corresponding quantum dimensions (2.5) with the braiding eigenvalues (2.14) are tabulated:

R_t	Highest weight	$\dim_q R_t$	λ_t	R_t	Highest weight	$\dim_q R_t$	λ_t
R_1	$\Lambda^{(N-2)} + 2\Lambda^{(1)}$	$\frac{[N-1][N-2][N+1][N+2]}{[2][2]}$	$-\lambda$	R_4	$\Lambda^{(N-1)} + \Lambda^{(1)}$	$[N-1][N+1];$	$\lambda^{3/2}$
R_2	$2\Lambda^{(N-1)} + \Lambda^{(2)}$	$\frac{[N-1][N-2][N+1][N+2]}{[2][2]}$	$-\lambda$	R_5	$2\Lambda^{(N)}$	1	λ^2
R_3	$\Lambda^{(N-2)} + \Lambda^{(2)}$	$\frac{[N]^2[N-3][N+1]}{[2][2]}$	$q\lambda$	R_6	$2\Lambda^{(N-1)} + 2\Lambda^{(1)}$	$\frac{[N]^2[N+3][N-1]}{[2][2]}$	$q^{-1}\lambda$
				R_7	$\Lambda^{(N-1)} + \Lambda^{(1)}$	$[N-1][N+1]$	$-\lambda^{3/2}$

Substituting the tabulated data in eqn.(2.13), the knot invariants for the framed knot $\mathcal{K}_{2m+1}^{(p)}$ carrying representation $\rho_0 = (\square, \square)$ can be computed. We have presented the tensor products for other composite representations in the appendix A. This data will be very useful to directly compute the composite invariants of framed knots (2.13) and framed links (2.16). These results are new and they are very essential to verify generalised Rudolph theorem (1.6) for many knots and links. The composite invariants also play a crucial role in verifying Marino's conjecture and obtaining the topological string amplitudes corresponding to cross-caps $c = 0, 1$ and 2 .

Using these $U(N)$ composite invariants and the $SO(N)$ invariants in appendix A of Ref. [21], we have verified generalised Rudolph's theorem (1.6). In the next section we will recapitulate the essential ideas of Marino's proposal for obtaining the cross-cap $c = 0$ and $c = 2$ topological amplitudes.

3 Reformulated Link Invariants

We will now review the conjectures proposed by Marino [2] for the reformulated $SO(N)$ invariants of knots and links. Particularly, we have to get the untwisted sector (oriented) contribution (1.4) to the open topological string amplitudes on the orientifold of the resolved conifold geometry.

Using the properties satisfied by Schur polynomials, eqn.(1.5) implies that the oriented invariants $\mathcal{R}_{R_1, \dots, R_r}[L]$ of the link L whose components $\mathcal{K}_1, \dots, \mathcal{K}_r$ are colored by representations R_1, \dots, R_r is given by

$$\mathcal{R}_{R_1, \dots, R_r}[L] = \sum_{S_1, T_1, \dots, S_r, T_r} \prod_{\alpha=1}^r N_{S_\alpha, T_\alpha}^{R_\alpha} \mathcal{H}_{(S_1, T_1), \dots, (S_r, T_r)}[L] , \quad (3.1)$$

where $N_{S_\alpha, T_\alpha}^{R_\alpha}$ are the Littlewood-Richardson coefficients and $\mathcal{H}_{(S_1, T_1), \dots, (S_r, T_r)}[L]$ are composite invariants in $U(N)$ Chern-Simons gauge theory of the link whose components carry the composite representations $(S_1, T_1), \dots, (S_r, T_r)$ of $U(N)$. The generating functional giving the oriented contribution to the open topological string partition function (1.4) is defined as

$$\mathcal{Z}_{\mathcal{R}}(v_1, \dots, v_r) = \sum_{R_1, \dots, R_r} \mathcal{R}_{R_1, \dots, R_r}[L] \prod_{\alpha=1}^r s_{R_\alpha}(v_\alpha); \quad \mathcal{F}_{\mathcal{R}}(v_1, \dots, v_r) = \log \mathcal{Z}_{\mathcal{R}}(v_1, \dots, v_r) , \quad (3.2)$$

where $s_R(v)$ are the Schur polynomials. Also the generating functionals for those involving $SO(N)$ Chern-Simons invariants, $\mathcal{G}_{R_1, \dots, R_r}$, of a link L are defined as

$$\mathcal{Z}_{\mathcal{G}}(v_1, \dots, v_r) = \sum_{R_1, \dots, R_r} \mathcal{G}_{R_1, \dots, R_r}[L] \prod_{\alpha=1}^r s_{R_\alpha}(v_\alpha); \quad \mathcal{F}_{\mathcal{G}}(v_1, \dots, v_r) = \log \mathcal{Z}_{\mathcal{G}}(v_1, \dots, v_r) . \quad (3.3)$$

Marino [2] has conjectured a specific form for these generating functionals:

$$\mathcal{F}_{\mathcal{R}}(v_1, \dots, v_r) = \sum_{d=1}^{\infty} \sum_{R_1, \dots, R_r} h_{R_1, \dots, R_r}(q^d, \lambda^d) \prod_{\alpha=1}^r s_{R_{\alpha}}(v_{\alpha}^d), \quad (3.4)$$

and

$$\mathcal{F}_{\mathcal{G}}(v_1, \dots, v_r) - \frac{1}{2} \mathcal{F}_{\mathcal{R}}(v_1, \dots, v_r) = \sum_{d \text{ odd}} \sum_{R_1, \dots, R_r} g_{R_1, \dots, R_r}(q^d, \lambda^d) \prod_{\alpha=1}^r s_{R_{\alpha}}(v_{\alpha}^d), \quad (3.5)$$

where $h_{R_1, \dots, R_r}(q, \lambda)$ and $g_{R_1, \dots, R_r}(q, \lambda)$ are the reformulated polynomial invariants involving the $U(N)$ and $SO(N)$ Chern-Simons link invariants respectively. The reformulated invariants are polynomials in q and λ and conjectured to obey the following form

$$h_{R_1, \dots, R_r}(q, \lambda) \text{ or } g_{R_1, \dots, R_r}(q, \lambda) = \sum_{Q, s} \frac{1}{q^{1/2} - q^{-1/2}} \tilde{N}_{R_1, \dots, R_r, Q, s} q^s \lambda^Q, \quad (3.6)$$

where $\tilde{N}_{R_1, \dots, R_r, Q, s}$ are integers. Though we know that the reformulated invariants $f_R(q, \lambda)$ obtained from $U(N)$ invariants $\mathcal{H}_R[L]$ satisfies the conjecture (1.2), it is not at all obvious that the reformulated invariant $h_{R_1, \dots, R_r}(q, \lambda)$ corresponding to the oriented invariants (3.1) involving linear combination of $U(N)$ composite invariants must obey a similar conjectured form (3.6). We check few examples in section 4 to verify Marino's conjecture on the oriented reformulated invariants. These reformulated invariants are further refined using the following equations, in order to reveal the BPS structure

$$h_{R_1, \dots, R_r}(q, \lambda) = \sum_{S_1, \dots, S_r} M_{R_1, \dots, R_r; S_1, \dots, S_r} \hat{h}_{S_1, \dots, S_r}(q, \lambda), \quad (3.7)$$

$$g_{R_1, \dots, R_r}(q, \lambda) = \sum_{S_1, \dots, S_r} M_{R_1, \dots, R_r; S_1, \dots, S_r} \hat{g}_{S_1, \dots, S_r}(q, \lambda), \quad (3.8)$$

where

$$M_{R_1, \dots, R_r; S_1, \dots, S_r} = \sum_{T_1, \dots, T_r} \prod_{\alpha=1}^r C_{R_{\alpha} S_{\alpha} T_{\alpha}} S_{T_{\alpha}}(q), \quad (3.9)$$

$R_{\alpha}, S_{\alpha}, T_{\alpha}$ are representations of the symmetric group $S_{\ell_{\alpha}}$ which can be labelled by a Young-Tableau with a total of ℓ_{α} boxes and C_{RST} are the Clebsch-Gordan coefficients of the symmetric group. $S_R(q)$ is non-zero only for the hook representations. For a hook representation having $\ell - d$ boxes in the first row of Young tableau with total ℓ boxes, $S_R(q) = (-1)^d q^{-(\ell-1)/2+d}$. Marino [2] has conjectured that the refined reformulated invariants $\hat{h}_{R_1, \dots, R_r}(q, \lambda)$ and $\hat{g}_{R_1, \dots, R_r}(q, \lambda)$ should have the following structure:

$$\hat{h}_{R_1, \dots, R_r}(q, \lambda) = z^{r-2} \sum_{g \geq 0} \sum_Q N_{R_1, \dots, R_r, g, Q}^{c=0} z^{2g} \lambda^Q, \quad (3.10)$$

$$\hat{g}_{R_1, \dots, R_r}(q, \lambda) = z^{r-1} \sum_{g \geq 0} \sum_Q \left(N_{R_1, \dots, R_r, g, Q}^{c=1} z^{2g} \lambda^Q + N_{R_1, \dots, R_r, g, Q}^{c=2} z^{2g+1} \lambda^Q \right), \quad (3.11)$$

where $N_{R_1, \dots, R_r, g, Q}^{c=0}$, $N_{R_1, \dots, R_r, g, Q}^{c=1}$ and $N_{R_1, \dots, R_r, g, Q}^{c=2}$ are the BPS invariants corresponding to cross-caps $c = 0, 1$ and 2 respectively and the variable $z = q^{1/2} - q^{-1/2}$.

In the next section, we obtain the reformulated invariants and obtain the BPS integers coefficients for framed knots and framed two-component links.

4 Verification of Marino's conjectures

The composite polynomials which we computed with our proposed choice of framing factor obeys the conjectures of Marino. We will briefly present some examples in this section.

4.1 Computation of Oriented Invariants $h_{R_1, \dots, R_r}(q, \lambda)$ and BPS invariants $N_{R_1, \dots, R_r, g, Q}^{c=0}$

In this subsection, we list the reformulated oriented invariants and the corresponding BPS invariants for simple framed knots to verify the conjecture (3.6).

Using the invariants for knots carrying composite representations as detailed in section 2 and appendix A, it is straightforward to obtain the reformulated oriented invariants. We have checked that these invariants for the torus knots and two-component links obey Marino's conjectured form. For completeness, we shall present some of the oriented reformulated invariants (3.6) for the unknot with framing p (U_p):

$$h_{\square\square}[U_p] = \frac{-\lambda^{p-1}}{(-1+q)^2(1+q)} \left(\lambda - 2q^{1+p}(-1+\lambda)(-1+q\lambda) + (-1)^p(-1+q)q(-1+\lambda^2) \right. \\ \left. + q(1+q+(-3+(-3+q)q)\lambda + (1+q)\lambda^2) \right) \quad (4.1)$$

$$h_{\square}[U_p] = \frac{\lambda^{p-1}}{(-1+q)^2(1+q)} \left(-2q^{1-p}(q-\lambda)(-1+\lambda) + (-1)^p(-1+q)q(-1+\lambda^2) \right. \\ \left. - (1+q)(\lambda+q(1+\lambda(-4+q+\lambda))) \right) \quad (4.2)$$

These results alongwith eqs.(3.7) and (3.10) give the the integer BPS invariants. For unknot with framing $p = 2$, the integers BPS invariants are

g	Q=1/2	3/2
0	-2	2

$N_{\square, g, Q}^{c=0}$

g	Q=2	3
0	3	-2

$N_{\square\square, g, Q}^{c=0}$

g	Q=1	2	3
0	-2	7	-4
1	0	2	-2

$N_{\square\square\square, g, Q}^{c=0}$

4.2 Computation of $SO(N)$ reformulated invariants $g_{R_1, R_2, \dots, R_r}(q, \lambda)$

We have computed the functions $g_{R_1, \dots, R_r}(q, \lambda)$ for framed unknot U_p , some framed torus knots $(\mathcal{K}_3^{(p)}, \mathcal{K}_5^{(p)})$, framed Hopf link $(H^{(p_1, p_2)})$ and the connected sum of two framed knots $\mathcal{K}_1 \# \mathcal{K}_2$. All these invariants obey the conjectured form (3.6). For completeness, we present g_R invariant for framed unknot U_p for few representations:

$$g_{\square\square} = \frac{\lambda^{3p/2-1}}{(-1+q)^2 (1+q)} ((-1)^p q (-1+q^p) (-1+\lambda) (1+q - q^p - q^{2p} - \lambda - q\lambda + q^{1+p}\lambda + q^{1+2p}\lambda)) \quad (4.3)$$

$$g_{\square\Box} = \frac{-\lambda^{3p/2-1}}{(-1+q)^2 (1+q)} (-1)^p q^{1-p} (-1+q^p) (-1+\lambda) (1-q(-2+\lambda) + q^p(-2+\lambda) - 2\lambda + q^{1+p}(-1+2\lambda)) \quad (4.4)$$

$$g_{\square\Box} = \frac{\lambda^{3p/2-1}}{(-1+q)^2 (1+q)} (-1)^p q^{\frac{1}{2}-3p} (-1+\lambda) (-q^{\frac{3}{2}} + q^{\frac{1}{2}+2p}(1-2\lambda) - q^{\frac{3}{2}+2p}(-2+\lambda) + q^{\frac{1}{2}+3p}(-1+\lambda) + q^{\frac{3}{2}+3p}(-1+\lambda) + \sqrt{q}\lambda) \quad (4.5)$$

Substituting values for p , the above equations reduce to the conjectured form (3.6).

4.3 $N_{(R_1, \dots, R_r), g, Q}^{c=1}$ and $N_{(R_1, \dots, R_r), g, Q}^{c=2}$ Computation

We have computed the integer coefficients corresponding to cross-cap $c = 1$ and $c = 2$ unoriented open string amplitude for various framed knots and framed links using eqns.(3.8, 3.11). The $c = 1$ BPS integers exactly matches our earlier paper results [21]. Both $c = 1$ and $c = 2$ integer BPS coefficients for torus knots with $p = 0$ framing agrees with the results in Refs. [2, 24]. We present the $c = 2$ BPS integer coefficients for few framed knots:

g	Q= 4	5	6	7
0	21	-63	63	-21
1	70	-231	231	-70
2	84	-322	322	-84
3	45	-219	219	-45
4	11	-78	79	-11
5	1	-14	14	-1
6	0	-1	1	0

$N_{\square\square, g, Q}^{c=2}$ for knot $\mathcal{K}_3^{(1)}$

g	Q= 4	5	6	7
0	28	-84	84	-28
1	126	-406	406	-126
2	210	-756	756	-210
3	165	-705	705	-165
4	66	-363	363	-66
5	13	-105	105	-13
6	1	-16	16	-1
7	0	-1	1	0

$N_{\square\Box, g, Q}^{c=2}$ for knot $\mathcal{K}_3^{(1)}$

g	Q=6	8	9	11
0	55	-275	275	-55
1	495	-2750	2750	-495
2	1716	-11110	11110	-1716
3	3003	-24090	24090	-3003
4	3003	-31746	31746	-3003
5	1820	-27118	27118	-1820
6	680	-15503	15503	-680
7	153	-5985	5985	-153
8	19	-1540	1540	-19
9	1	-253	253	-1
10	0	-24	24	0
11	0	-1	1	0

$N_{\square\square,g,Q}^{c=2}$ for knot $\mathcal{K}_5^{(1)}$

g	Q=6	8	9	11
0	66	-330	330	-66
1	715	-3905	3905	-715
2	3003	-18656	18656	-3003
3	6435	-47905	47905	-6435
4	8008	-75218	75218	-8008
5	6188	-77415	77415	-6188
6	3060	-54248	54248	-3060
7	969	-26333	26333	-969
8	190	-8855	8855	-190
9	21	-2024	2024	-21
10	1	-300	300	-1
11	0	-26	26	0
12	0	-1	1	0

$N_{\square\square,g,Q}^{c=2}$ for knot $\mathcal{K}_5^{(1)}$

For the connected sum of trefoil with trefoil ($\mathcal{K}_3^{(0)} \# \mathcal{K}_3^{(0)}$), we have computed the BPS integers. For $\square\square$ representation, the integers $N_{\square\square,g,Q}^{c=2}$ are

g	Q=4	5	6	7	8	9	10	11
0	-46	627	-2210	3524	-2891	1190	-193	-1
1	-115	2857	-12709	23835	-22244	10068	-1517	-121
2	-114	5764	-32974	73721	-78050	37511	-4648	-1210
3	-54	6412	-48952	133320	-160443	79607	-5171	-4719
4	-12	4241	-45575	155369	-214257	107758	1914	-9438
5	-1	1707	-27770	122272	-195972	98868	11907	-11011
6	0	410	-11234	66279	-126105	63513	15145	-8008
7	0	54	-2987	24753	-57626	28938	10608	-3740
8	0	3	-501	6247	-18593	9314	4652	-1122
9	0	0	-48	1016	-4138	2070	1309	-209
10	0	0	-2	96	-604	302	230	-22
11	0	0	0	4	-52	26	23	-1
12	0	0	0	0	-2	1	1	0

5 Summary and Discussions

We have explicitly demonstrated the direct evaluation of invariants of some framed knots and links carrying composite representations in $U(N)$ Chern-Simons gauge theory. Particularly, we

proposed a specific choice for the $U(1)$ charge corresponding to the composite representations (2.10) so that the composite invariants for framed knots and links are polynomials in variables q, λ . Further, this direct method enabled us to verify generalised Rudolph's theorem for many framed knots (1.6).

The composite invariants was very essential to obtain the untwisted sector open topological string amplitude (3.4) on the orientifold of the resolved conifold geometry. Similar to Ooguri-Vafa conjecture (1.2), Marino [2] conjectured a form for the reformulated invariants (3.6) and the refined reformulated invariants (3.10). We have verified the conjecture for many framed knots and links and presented the reformulated invariants for few examples. The cross-cap $c = 0$ BPS integer coefficients (3.10) are also tabulated for these examples.

In earlier works [19–21], there was difficulty in separating $c = 0$ and $c = 2$ contribution from the topological string free energy (1.4) but using the parity argument in variable $\sqrt{\lambda}$, the cross-cap $c = 1$ amplitude could be determined. With the present work on composite invariants following the approach [2], we can determine the unoriented topological string amplitude (1.4) by subtracting the untwisted sector contribution from the free energy of the open topological string theory on the orientifold. We have checked that the reformulated SO invariants obtained from the unoriented topological string free energy also obeys Marino's conjectured form (3.6). Further, the refined SO reformulated invariants obtained using eqn. (3.8) satisfies the conjectured form (3.11). We have tabulated the BPS integer invariants corresponding to cross-cap $c = 2$ obtained from reformulated invariants (3.11) for some framed knots. The $c = 1$ integer coefficients agrees with our earlier work [21]. The BPS integer coefficients for the standard framing ($p = 0$) torus knots and torus links agrees with the results in Ref. [24]. The verification of Marino's conjectures for many framed knots and two-component framed links indirectly confirms that our choice of the $U(1)$ charge (2.10) for the composite representations is correct.

The Marino's conjectures, which we verified for some torus knots and torus links, should be obeyed by non-torus knots and non-torus links as well. The Chern-Simons approach requires the $SU(N)$ quantum Racah coefficients for the non-torus knot invariant evaluation. Unfortunately, these coefficients are not available in the literature. In Ref. [22], the $SU(N)$ quantum Racah coefficients for some representations could be determined using isotopy equivalence of knots enabling evaluation of non-torus knot invariants. We believe that there must be a similar approach of determining composite invariants for the non-torus knots.

It will be interesting to generalise these integrality properties in the context of Khovanov homology [28] and Kauffman homology [29]. We hope to report on this work in a future publication.

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Appendix

A $U(N)$ Composite Knot Invariants

- For the composite representation $\rho_{02} \equiv (\square, \square)$ whose highest weight is $\Lambda^{(N-1)} + 2\Lambda^{(1)}$, the highest weights and the braiding eigenvalues corresponding to the irreducible representations $R_t \in \rho_{02} \otimes \rho_{02}$ are

R_t	highest weight	λ_t	R_t	highest weight	λ_t
R_1	$2\Lambda^{(N-1)} + 4\Lambda^{(1)}$	$q^{-3/2}\lambda^{3/2}$	R_7	$2\Lambda^{(N-1)} + 2\Lambda^{(2)}$	$q^{3/2}\lambda^{3/2}$
R_2	$\Lambda^{(N)} + \Lambda^{(N-2)} + 4\Lambda^{(1)}$	$-q^{-1/2}\lambda^{3/2}$	R_8	$\Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(2)}$	$-q^{5/2}\lambda^{3/2}$
R_3	$2\Lambda^{(N-1)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	$-q^{1/2}\lambda^{3/2}$	R_9	$2\Lambda^{(N)} + 2\Lambda^{(1)}$	$q^{3/2}\lambda^{5/2}\lambda$
R_4	$\Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	$q^{3/2}\lambda^{3/2}$	R_{10}	$2\Lambda^{(N)} + \Lambda^{(2)}$	$-q^{5/2}\lambda^{5/2}$
R_5	$\Lambda^{(N)} + \Lambda^{(N-1)} + 3\Lambda^{(1)}$	$q^{1/2}\lambda^2$	R_{11}	$\Lambda^{(N)} + \Lambda^{(N-1)} + 3\Lambda^{(1)}$	$-q^{1/2}\lambda^2$
R_6	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	$q^2\lambda^2$	R_{12}	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	$-q^2\lambda^2$

Using the above table, we can evaluate directly the composite invariants for framed torus knots and links obtained as closure of two strand braids (2.13, 2.16).

- For the composite representation $\rho_{03} \equiv (\square, \square)$ whose highest weight is $\Lambda^{(N-1)} + \Lambda^{(2)}$, the representations R_t obtained from $\rho_{03} \otimes \rho_{03}$ and the signs of the braiding eigenvalues: ϵ_{R_t} are

$$\begin{aligned}
R_1 &= 2\Lambda^{(N-1)} + 2\Lambda^{(2)}; \epsilon_1 = 1 & R_2 &= \Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(2)}; \epsilon_2 = -1 \\
R_3 &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_3 = 1 & R_4 &= 2\Lambda^{(N)} + 2\Lambda^{(1)}; \epsilon_4 = 1 \\
R_5 &= 2\Lambda^{(N-1)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_5 = -1 & R_6 &= \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_6 = 1 \\
R_7 &= 2\Lambda^{(N)} + \Lambda^{(2)}; \epsilon_7 = -1 & R_8 &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(3)}; \epsilon_8 = 1 \\
R_9 &= \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_9 = 1 & R_{10} &= \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_{10} = -1 \\
R_{11} &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{11} = -1 & R_{12} &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(3)}; \epsilon_{12} = -1
\end{aligned}$$

For the above irreducible representations, quadratic casimir and eigenvalues can be computed using eqns. (2.7, 2.14). With this data, the polynomials of the framed knots $\mathcal{H}_{\square, \square}[\mathcal{K}]$ and framed links carrying the composite representation (\square, \square) can be computed.

- The invariants of knots carrying the composite representation $\rho_{04} \equiv (\square, \square)$ with highest weight $2\Lambda^{(N-1)} + 2\Lambda^{(1)}$ will be useful for verifying generalised Rudolph theorem (1.6). The highest weights of the irreducible representations R_t obtained from $\rho_{04} \otimes \rho_{04}$ and the signs of the braiding eigenvalues ϵ_{R_t} are

R_t	Highest weight	ϵ_{R_t}	R_t	Highest weight	ϵ_{R_t}
R_1	$4\Lambda^{(N-1)} + 4\Lambda^{(1)}$	1	R_2	$\Lambda^{(N)} + 2\Lambda^{(N-1)} + \Lambda^{(N-2)} + 4\Lambda^{(1)}$	-1
R_3	$2\Lambda^{(N)} + 2\Lambda^{(N-2)} + 4\Lambda^{(1)}$	1	R_4	$\Lambda^{(N)} + 3\Lambda^{(N-1)} + 3\Lambda^{(1)}$	1
R_5	$2\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(N-2)} + 3\Lambda^{(1)}$	1	R_6	$\Lambda^{(N)} + 2\Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	1
R_7	$2\Lambda^{(N)} + 2\Lambda^{(N-2)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	-1	R_8	$2\Lambda^{(N)} + 2\Lambda^{(N-1)} + 2\Lambda^{(1)}$	1
R_9	$2\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}$	-1	R_{10}	$2\Lambda^{(N)} + 2\Lambda^{(N-2)} + 2\Lambda^{(2)}$	1
R_{11}	$2\Lambda^{(N)} + 2\Lambda^{(N-1)} + \Lambda^{(2)}$	-1	R_{12}	$3\Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}$	1
R_{13}	$3\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}$	1	R_{14}	$4\Lambda^{(N)}$	1
R_{15}	$4\Lambda^{(N-1)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	-1	R_{16}	$4\Lambda^{(N-1)} + 2\Lambda^{(2)}$	1
R_{17}	$\Lambda^{(N)} + 3\Lambda^{(N-1)} + 3\Lambda^{(1)}$	-1	R_{18}	$2\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(N-2)} + 3\Lambda^{(1)}$	-1
R_{19}	$\Lambda^{(N)} + 3\Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	1	R_{20}	$\Lambda^{(N)} + 3\Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	-1
R_{21}	$\Lambda^{(N)} + 2\Lambda^{(N-1)} + \Lambda^{(N-2)} + 2\Lambda^{(2)}$	-1	R_{22}	$2\Lambda^{(N)} + 2\Lambda^{(N-1)} + 2\Lambda^{(1)}$	1
R_{23}	$2\Lambda^{(N)} + 2\Lambda^{(N-1)} + 2\Lambda^{(1)}$	-1	R_{24}	$2\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}$	1
R_{25}	$3\Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(1)}$	-1	R_{26}	$3\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}$	-1

- For another composite representation $\rho_{05} \equiv (\square, \square)$ whose highest weight is $\Lambda^{(N-2)} + \Lambda^{(2)}$, the irreducible representations R_t obtained from $\rho_{05} \otimes \rho_{05}$ and the signs of the braiding eigenvalues ϵ_t are

$$\begin{aligned}
R_1 &= \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_1 = 1 & R_2 &= \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_2 = -1 \\
R_3 &= \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_3 = 1 & R_4 &= \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_4 = 1 \\
R_5 &= \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_5 = 1 & R_6 &= \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(1)} + \Lambda^{(1)}; \epsilon_6 = -1 \\
R_7 &= \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_7 = -1 & R_8 &= \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_8 = 1 \\
R_9 &= \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_9 = -1 & R_{10} &= \Lambda^{(N-1)} + \Lambda^{(N-1)} + \Lambda^{(2)}; \epsilon_{10} = -1 \\
R_{11} &= \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{11} = 1 & R_{12} &= \Lambda^{(N-1)} + \Lambda^{(N-1)} + \Lambda^{(1)} + \Lambda^{(1)}; \epsilon_{12} = 1 \\
R_{13} &= \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(3)}; \epsilon_{13} = 1 & R_{14} &= \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(3)}; \epsilon_{14} = 1 \\
R_{15} &= \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(4)}; \epsilon_{15} = 1 & R_{16} &= \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_{16} = -1 \\
R_{17} &= \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(4)}; \epsilon_{17} = 1 & R_{18} &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}; \epsilon_{18} = 1 \\
R_{19} &= 2\Lambda^{(N)}; \epsilon_{19} = 1 & R_{20} &= \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{20} = -1 \\
R_{21} &= \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{21} = -1 & R_{22} &= \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{22} = -1 \\
R_{23} &= \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{23} = 1 & R_{24} &= \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(3)}; \epsilon_{24} = -1 \\
R_{25} &= \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(3)}; \epsilon_{25} = -1 & R_{26} &= \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}; \epsilon_{26} = -1
\end{aligned}$$

The composite invariants $\mathcal{H}_{(R,R)}[\mathcal{K}]$ for $R = \square$ and $R = \square$ and the corresponding $SO(N)$ invariants $\mathcal{G}_R[\mathcal{K}]$ given in appendix A of Ref. [21] satisfy the generalised Rudolph theorem (1.6).

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